

k -STEP NILPOTENT LIE ALGEBRAS

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ABSTRACT. The classification of complex of real finite dimensional Lie algebras which are not semi simple is still in its early stages. For example the nilpotent Lie algebras are classified only up to the dimension 7. Moreover, to recognize a given Lie algebra in a classification list is not so easy. In this work we propose a different approach to this problem. We determine families for some fixed invariants, the classification follows by a deformation process or contraction process. We focus on the case of 2 and 3-step nilpotent Lie algebras. We describe in both cases a deformation cohomology of this type of algebras and the algebras which are rigid regarding this cohomology. Other p -step nilpotent Lie algebras are obtained by contraction of the rigid ones.

1. INTRODUCTION

Let \mathfrak{g} be a Lie algebra over an algebraically closed field \mathbb{K} of characteristic 0. We denote by $\{\mathfrak{g}^k \mid k = 0, 1, 2, \dots\}$ the descending central series of \mathfrak{g} defined by $\mathfrak{g}^0 = \mathfrak{g}$ and $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$, for $k = 1, 2, \dots$. In any Lie algebra the lower central series is a *filtration* in the sense that $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j+1}$. One calls \mathfrak{g} *nilpotent* if there is an integer n such that $\mathfrak{g}^n = \{0\}$. If moreover $\mathfrak{g}^{n-1} \neq \{0\}$ then n is called the *nilpotent index* or *nilindex* of \mathfrak{g} and denoted by $n(\mathfrak{g})$. A nilpotent Lie algebra \mathfrak{g} with nilindex $n(\mathfrak{g})$ is called $n(\mathfrak{g})$ -step nilpotent. From Engel Theorem, each adjoint operator $ad(X)$ is nilpotent with $ad(X)^{n(\mathfrak{g})} = 0$ for any $X \in \mathfrak{g}$. In the present work, we regard a p -step nilpotent Lie algebra as a Lie algebra whose Lie bracket satisfies a p -associativity condition. We define for p equal to 2 or 3 new cohomological spaces whose main property is that the second group of cohomology characterizes the deformations of p -step nilpotent Lie algebras in the class of p -step nilpotent Lie algebras. Recall that if we denote by $H_C^*(\mathfrak{g}, \mathfrak{g})$ the Chevalley-Eilenberg cohomological spaces of \mathfrak{g} , the space $H_C^2(\mathfrak{g}, \mathfrak{g})$ parametrizes the deformations of \mathfrak{g} but in general such a deformation is not nilpotent. Cohomological spaces also different from the Chevalley-Eilenberg's ones are defined in [19] and more adapted to nilpotent Lie algebras. In fact, if p is fixed, the 2-cocycles determine p -step nilpotent cocycles of $\mathfrak{g}/\mathfrak{g}^{p+1}$ by $\{X \in \mathfrak{g}, ad(Y)^p(X) = 0 \ \forall Y \in \mathfrak{g}\}$. Our point of view is a little bit different. We will describe cohomological spaces which are associated to deformations of p -step nilpotent Lie algebras is this family of p -step nilpotent Lie algebras.

Let $pNil_n$ be the variety of n -dimensional p -step nilpotent Lie algebras over \mathbb{K} . Recall that the characteristic sequence $c(\mathfrak{g})$ of a nilpotent Lie algebra is the invariant, up an isomorphism, defined as follows (see [1, 4]): if X is in \mathfrak{g} then the linear operator $ad(X)$ is nilpotent and we denote by $c(X)$ the decreasing sequence of the dimensions of Jordan blocks of $ad(X)$. The characteristic sequence of \mathfrak{g} is

$$c(\mathfrak{g}) = \inf\{c(X), X \in \mathfrak{g} - \mathfrak{g}^1\},$$

considering the lexicographic order. If $\mathfrak{g} \in p\text{Nil}p_n$, then $c(\mathfrak{g}) = (p, \dots, p, p-1, \dots, 1)$ where p appears k_p times, $p-1$ appears k_{p-1} times, \dots , and 1 appears k_1 times with $k_p \neq 0 \neq k_1$ and $n = \sum_{i=1}^p ik_i$. It is clear that if $\mathfrak{g}_1 \in p\text{Nil}p_n$ is a formal deformation of $\mathfrak{g} \in p\text{Nil}p_n$ in the Gerstenhaber sense ([2, 4, 10]) then $c(\mathfrak{g}) \leq c(\mathfrak{g}_1)$. If the characteristic sequence of the Lie algebra \mathfrak{g} is $c(\mathfrak{g}) = (p, \dots, p, l, 1)$ with $l \equiv n-1 \pmod{p}$ then any deformation $\mathfrak{g}_1 \in p\text{Nil}p_n$ of \mathfrak{g} has the same characteristic sequence. We denote by $\mathcal{F}_{k_p, k_{p-1}, \dots, k_1}^{n,p}$ the family of n -dimensional p -step nilpotent Lie algebras with characteristic sequence $(p, \dots, p, p-1, \dots, 1)$ where i appears k_i times, for $1 \leq i \leq p$. The previous remark shows that $\mathcal{F}_{k_p, k_l=1, k_1=1}^{pk_p+l+1, p}$ is an open set in $p\text{Nil}p_n$. The aim of this paper is to study this open set for $p = 2$ and 3.

The study of this family is interesting, at least, for two reasons:

- (1) The important works of Vergne on nilpotent Lie algebras lead to conjecture that "there exist no rigid nilpotent Lie algebras (in the variety of Lie algebras)", where rigidity means that any deformation is isomorphic. This conjecture has still not been settled. In the present work, we define a notion of rigidity on $p\text{Nil}p_n$ associated with deformations internal to this variety. If there exists a counterexample to the Vergne's conjecture, necessarily this algebra will also be rigid in $p\text{Nil}p_n$ for a good p .
- (2) The classification of nilpotent Lie algebras is still an open problem. General results are known for the dimensions smaller than or equal to 7 [3] and partial results for the dimension 8 [18]. In the present work we determine rigid Lie algebras in $\mathcal{F}_{k_p, 0, \dots, 0, k_i=1, 0, \dots, k_1=1}^{n,p}$ for $p \leq 3$. Then the classification of the elements of this family is in part determined by the contractions of the rigid Lie algebras contained in this variety.

We give also some examples of study of families $\mathcal{F}_{k_p, k_{p-1}, \dots, k_1}^{n,p}$ associated to p -step nilpotent Lie algebras with a characteristic sequence which is not maximal, in order to show how it can be useful to classify the $p\text{Nil}p_n$.

Notations. If \mathfrak{g} is a Lie algebra over \mathbb{K} with Lie bracket μ , we denote by $H_C^*(\mathfrak{g}, \mathfrak{g})$ or $H_C^*(\mu, \mu)$ its Chevalley-Eilenberg cohomology. The corresponding coboundary operator will be denoted $\delta_{C, \mu}$, or when no confusion is possible, δ_C .

2. ASSOCIATIVE LIE MULTIPLICATION AND 2-STEP NILPOTENT LIE ALGEBRAS

2.1. The variety $2\text{Nil}p_n$. Let \mathfrak{g} be a Lie algebra over an algebraic field \mathbb{K} of characteristic 0. If we denote by $[X, Y]$ the Lie bracket of \mathfrak{g} , it satisfies the following identities

$$\begin{cases} [X, Y] = -[Y, X], \\ [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0 \quad (\text{Jacobi Identity}), \end{cases}$$

for any $X, Y, Z \in \mathfrak{g}$. We assume moreover that the Lie bracket is also an associative product, that is, it satisfies

$$[[X, Y], Z] = [X, [Y, Z]],$$

for any $X, Y, Z \in \mathfrak{g}$. The Jacobi identity therefore implies

$$(1) \quad [[Z, X], Y] = 0.$$

Proposition 1. *The Lie bracket of the Lie algebra \mathfrak{g} is an associative product if and only if \mathfrak{g} is a 2-step nilpotent Lie algebra.*

In fact, the relation $[[Z, X], Y] = 0$ means that $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = \mathcal{C}^2(\mathfrak{g}) = 0$ where $\mathcal{C}^i(\mathfrak{g})$ denotes the ideals of the descending central series of \mathfrak{g} . The converse is obvious.

Let n be a fixed integer. Any n -dimensional \mathbb{K} -Lie algebra \mathfrak{g} is identified with the pair (\mathbb{K}^n, μ) where μ is the Lie bracket on the underlying vector space of \mathfrak{g} , that is \mathbb{K}^n . We fix a basis of $\{e_i\}$ of \mathbb{K}^n . Then μ is determined by its structures constants $\mu(e_i, e_j) = \sum_{k=1}^n C_{ij}^k e_k$.

Equation (1) is equivalent to

$$(2) \quad \begin{cases} C_{ij}^k = -C_{ji}^k, \\ \sum_{l=1}^n C_{ij}^l C_{lk}^s = 0, \end{cases} \quad \forall k, s \in \{1, \dots, n\}, 1 \leq i < j \leq n.$$

These polynomial identities define an algebraic variety on \mathbb{K}^{n^3} denoted $2Nilp_n$ in which any point is a 2-step n -dimensional nilpotent Lie algebra over \mathbb{K} .

This variety is fibered by the orbits of the action of $GL(n, \mathbb{K})$ on $2Nilp_n$

$$GL(n, \mathbb{K}) \times 2Nilp_n \rightarrow 2Nilp_n$$

given by $(f, \mu) \mapsto f \cdot \mu$ with $f \cdot \mu(X, Y) = f^{-1} \mu(f(X), f(Y))$. The orbit $\theta(\mu)$ of μ is the set of Lie algebras isomorphic to $\mathfrak{g} = (\mathbb{K}^n, \mu)$.

Definition 2. A Lie algebra $\mu \in 2Nilp_n$ is called rigid in $2Nilp_n$ if its orbit $\theta(\mu)$ is open in $2Nilp_n$.

This notion of rigidity is equivalent to say that any formal deformation $\mu_t = \mu + \sum t^i \phi_i$ of μ in $2Nilp_n$ is isomorphic to μ (For general definition of formal deformations see [2, 13, 4]). We know that there exists a cohomological sequence $H_{def}^*(\mu, \mu)$ which parametrizes the deformations of μ . We shall define this cohomology.

2.2. The CH-cohomology of 2-step nilpotent Lie algebra. Let $\mathfrak{g}_0 = (\mathbb{K}^n, \mu_0)$ be a 2-step nilpotent Lie algebra over \mathbb{K} . The CH-cohomology of \mathfrak{g}_0 is the cohomology associated with the complex $(\mathcal{C}^n(\mathbb{K}^n, \mathbb{K}^n), \delta_{CH, \mu_0}^n)$ where $\mathcal{C}^n(\mathbb{K}^n, \mathbb{K}^n)$ is the vector space of skew n -linear maps on \mathfrak{g}_0 with values in \mathfrak{g}_0 and $\delta_{CH, \mu_0}^n : \mathcal{C}^n(\mathbb{K}^n, \mathbb{K}^n) \rightarrow \mathcal{C}^{n+1}(\mathbb{K}^n, \mathbb{K}^n)$ is the linear operator defined by

$$\begin{cases} \delta_{CH, \mu_0}^{2p} \psi(X_1, \dots, X_{2p+1}) = \mu_0(X_1, \psi(X_2, \dots, X_{2p+1})) \\ \quad + \sum_{i=1}^p \psi(X_1, \dots, \mu_0(X_{2i}, X_{2i+1}), \dots, X_{2p+1}), \\ \delta_{CH, \mu_0}^{2p-1} \psi(X_1, \dots, X_{2p}) = \mu_0(X_1, \psi(X_2, \dots, X_{2p})) \\ \quad + \sum_{i=1}^{p-1} \psi(X_1, \dots, \mu_0(X_{2i+1}, X_{2i+2}), \dots, X_{2p}). \end{cases}$$

Since $\delta_{CH, \mu_0}^n \circ \delta_{CH, \mu_0}^{n+1} = 0$, we define

$$H_{CH}^n(\mu_0, \mu_0) = \frac{Z_{CH}^n(\mu_0, \mu_0)}{B_{CH}^n(\mu_0, \mu_0)}$$

with $Z_{CH}^n(\mu_0, \mu_0) = \{\varphi \in \mathcal{C}^n(\mathbb{K}^n, \mathbb{K}^n), \delta_{CH, \mu_0}^n \varphi = 0\}$ and $B_{CH}^n(\mu_0, \mu_0) = \text{Im} \delta_{CH, \mu_0}^{n-1}$.

We shall prove that this cohomology is the cohomology of deformations for 2-step nilpotent Lie algebras. For this, we introduce a quadratic operad with the property, any algebra over this operad is a 2-step nilpotent Lie algebra.

An operad is a sequence $\mathcal{P} = \{\mathcal{P}(n), n \in \mathbb{N}^*\}$ of $\mathbb{K}[\Sigma_n]$ -modules, where $\mathbb{K}[\Sigma_n]$ is the algebra group associated with the symmetric group Σ_n , with $comp_i$ -operations (see [14]). The main example corresponds to the free operad $\Gamma(E) = \{\Gamma(E)(n)\}$ generated by a $\mathbb{K}[\Sigma_2]$ -module E . An operad \mathcal{P} is called binary quadratic if there is a $\mathbb{K}[\Sigma_2]$ -module E and a $\mathbb{K}[\Sigma_3]$ -submodule R of $\Gamma(E)(3)$ such that \mathcal{P} is isomorphic to $\Gamma(E)/\mathcal{R}$ where \mathcal{R} is the operadic ideal generated by $\mathcal{R}(3) = R$.

Proposition 3. *There exists a binary quadratic operad, denoted by $2\mathcal{N}ilp$, with the property that any $2\mathcal{N}ilp$ -algebra is a 2-step nilpotent Lie algebra. Moreover, this operad is Koszul.*

In fact, we consider $E = sgn_2$, that is, the representation of Σ_2 by the signature, then $\Gamma(E)(3) = sgn_3 \oplus V_2$ where $V_2 = \{(x, y, z) \in \mathbb{K}^3, x + y + z = 0\}$. Let R be the submodule of $\Gamma(E)(3)$ generated by the vectors $(x_i \cdot x_j) \cdot x_k$, with i, j, k all different. We deduce that $2\mathcal{N}ilp(2)$ is the $\mathbb{K}[\Sigma_2]$ -module generated by $x_1 \cdot x_2$ with the relation $x_2 \cdot x_1 = -x_1 \cdot x_2$ so it is a 1-dimensional vector space and $2\mathcal{N}ilp(3) = \{0\}$. Let us prove that this operad satisfies the Koszulness property (see [14]). The generating function of a binary quadratic operad $2\mathcal{N}ilp$ is

$$g_{2\mathcal{N}ilp}(x) = \sum_{a \geq 1} \frac{1}{a!} \dim(2\mathcal{N}ilp(a)) x^a = x + \frac{x^2}{2}.$$

The dual operad $2\mathcal{N}ilp^!$ of the operad $2\mathcal{N}ilp$ is the quadratic operad $2\mathcal{N}ilp^! := \Gamma(E^\vee)/(R^\perp)$, where $R^\perp \subset \Gamma(E^\vee)(3)$ is the annihilator of $R \subset \Gamma(E)(3)$. Thus the dual operad $(2\mathcal{N}ilp)^!$ is $\Gamma(\mathbb{1})$ the free operad generated by a commutative operation. So

$$\dim(2\mathcal{N}ilp^!)(1) = 1, \dim(2\mathcal{N}ilp^!)(2) = 1, \dim(2\mathcal{N}ilp^!)(3) = 3, \dim(2\mathcal{N}ilp^!)(4) = 15$$

and more generally, if we denote by d_k the dimension of $(2\mathcal{N}ilp)^!(k)$, we have

$$\begin{cases} d_{2k+1} = \sum_{i=1}^k C_{2k+1}^i d_i d_{2k+1-i}, \\ d_{2k} = \sum_{i=1}^{k-1} C_{2k}^i d_i d_{2k-i} + \frac{1}{2} C_{2k}^k d_k^2. \end{cases}$$

So the generating function of $2\mathcal{N}ilp^!$ is

$$\sum_{k \geq 1} \frac{d_k}{k!} x^k.$$

If an operad \mathcal{P} is Koszul, then its dual $\mathcal{P}^!$ is also Koszul and the generating functions are related by the functional equation

$$g_{\mathcal{P}}(-g_{\mathcal{P}^!}(-x)) = x.$$

It is known that $\Gamma(\mathbb{1})$ is Koszul, so also $2\mathcal{N}ilp$ and this implies the proposition. We can verify that the generating function $g_{2\mathcal{N}ilp}$ of the operad $2\mathcal{N}ilp$ satisfies the functional equation

$$g_{2\mathcal{N}ilp}(-g_{2\mathcal{N}ilp^!}(-x)) = x.$$

The operadic cohomology of a quadratic operad is described in [14]. We find the CH-cohomology. Since $2\mathcal{N}ilp$ satisfies the Koszul property, this cohomology coincides with the deformation cohomology for $2\mathcal{N}ilp$ -algebras, that is the cohomology which parametrizes the formal deformations of a 2-step nilpotent Lie algebra.

Remark. Let us consider an associative algebra (A, \cdot) where $x \cdot y$ denotes the multiplication in A . Thus

$$[x, y] = x \cdot y - y \cdot x$$

is a Lie bracket. This Lie bracket is associative if and only if the multiplication of A satisfies

$$(x \cdot y) \cdot z - (y \cdot x) \cdot z - (z \cdot x) \cdot y + (z \cdot y) \cdot x = 0.$$

This kind of nonassociative algebra belong to the family of nonassociative algebras described in [9]. In this work, we define “natural” non-associative algebras, denoted G -associative. Thus we can look what happen when we consider a Lie algebra whose Lie bracket satisfies such non-associative identities. But, we can look very quickly that if a Lie bracket is G -associative, then it is also associative and we return to the initial case.

2.3. Deformations and Rigidity in $2\mathcal{N}ilp_n$. Let $\mathfrak{g}_0 = (\mathbb{K}^n, \mu_0)$ be a n -dimensional 2-step nilpotent Lie algebra over \mathbb{K} . Let $\mathfrak{g} = (\mathbb{K}^n, \mu)$ be a deformation of \mathfrak{g}_0 in $2\mathcal{N}ilp_n$, that is

$$\mu = \mu_0 + \sum t^i \varphi_i$$

where φ_i are skew-symmetric bilinear maps on \mathbb{K}^n . The Jacobi condition related to μ gives in particular that $\varphi_1 \in Z_{CH}^2(\mathfrak{g}_0, \mathfrak{g}_0)$ and

$$(3) \quad \begin{cases} \delta_{CH, \mu_0} \varphi_1 = 0, \\ \varphi_1 \bullet \varphi_1 = 0, \end{cases}$$

where $\varphi \bullet \varphi = 0$ is the Jacobi identity. A deformation of type $\mu = \mu_0 + t\varphi$ will be called a linear deformation.

Since $H_{CH}^*(\mathfrak{g}_0, \mathfrak{g}_0)$ parametrizes the deformations of \mathfrak{g}_0 in $2\mathcal{N}ilp_n$, Nijenhuis-Richardson Theorem ([16]) gives:

Theorem 4. *Let \mathfrak{g}_0 be a n -dimensional 2-step nilpotent Lie algebra on \mathbb{K} . If $H_{CH}^2(\mathfrak{g}_0, \mathfrak{g}_0) = 0$, then \mathfrak{g}_0 is rigid in $2\mathcal{N}ilp_n$.*

Proposition 5. [11] *The $(2p+1)$ -dimensional Heisenberg algebra \mathfrak{h}_{2p+1} is rigid in $2\mathcal{N}ilp_{2p+1}$.*

Proof. The dimension of the algebra of derivations of \mathfrak{h}_{2p+1} is computed in [7]. We deduce $\dim B_{CH}^2(\mathfrak{h}_{2p+1}, \mathfrak{h}_{2p+1}) = p(2p+1)$. Let $\{X_1, \dots, X_{2p+1}\}$ be the basis of \mathfrak{h}_{2p+1} satisfying

$$[X_1, X_2] = \dots = [X_{2i-1}, X_{2i}] = \dots = [X_{2p-1}, X_{2p}] = X_{2p+1}.$$

If $\varphi(X_i, X_j) = \sum_{k=1}^{2p+1} a_{ij}^k X_k$, then $\delta_{CH, \mu_0}(\varphi)(X_i, X_j, X_k) = 0$ is equivalent to

$$\varphi(\mu_0(X_i, X_j), X_k) + \mu_0(\varphi(X_i, X_j), X_k) = 0,$$

that is,

$$\left\{ \begin{array}{l} \varphi(X_1, X_2) = \sum_{k=1}^{2p+1} a_{12}^k X_k; \\ \varphi(X_1, X_i) = a_{1i}^{2p+1} X_{2p+1}, 3 \leq i \leq 2p; \quad \varphi(X_1, X_{2p+1}) = -a_{12}^1 X_{2p+1}; \\ \varphi(X_2, X_i) = a_{2i}^{2p+1} X_{2p+1}, 3 \leq i \leq 2p; \quad \varphi(X_2, X_{2p+1}) = a_{12}^1 X_{2p+1}; \\ \dots \\ \varphi(X_{2i-1}, X_{2i}) = \sum_{k=1}^{2p} a_{12}^k X_k + a_{2i-1, 2i}^{2p+1} X_{2p+1}, 2 \leq i \leq p; \\ \varphi(X_l, X_s) = a_{ls}^{2p+1} X_{2p+1}, (l, s) \neq (2i-1, 2i); \\ \varphi(X_{2l}, X_{2p+1}) = a_{12}^{2l-1} X_{2p+1}, l \leq p; \\ \varphi(X_{2l-1}, X_{2p+1}) = -a_{12}^{2l} X_{2p+1}, l \leq p. \end{array} \right.$$

We deduce that $\dim Z_{CH}^2(\mathfrak{h}_{2p+1}, \mathfrak{h}_{2p+1}) = p(2p+1)$. So $\dim H_{CH}^2(\mathfrak{h}_{2p+1}, \mathfrak{h}_{2p+1}) = 0$ and \mathfrak{h}_{2p+1} is rigid in $2Nilp_{2p+1}$.

Let us note that \mathfrak{h}_{2p+1} is not rigid in the variety of $(2p+1)$ -dimensional Lie algebras. From [11], any deformation of \mathfrak{h}_{2p+1} is a contact Lie algebra. The rigidity in $2Nilp_n$ can be understood by the fact that any nilpotent Lie algebra whose characteristic sequence is $(2, 1, \dots, 1)$ is isomorphic to a direct product of an Heisenberg algebra by an abelian algebra.

2.4. 2-step nilpotent Lie algebras with characteristic sequence $(2, \dots, 2, 1)$. Let $\mathfrak{g}_{p,1}$ be the $(2p+1)$ -dimensional Lie algebras defined by the following brackets given in the basis $\{X_1, \dots, X_{2p+1}\}$ by

$$[X_1, X_{2i}] = X_{2i+1}, \quad 1 \leq i \leq p,$$

the other non defined brackets are equal to zero. Its characteristic sequence is $(2, \dots, 2, 1)$ where 2 appears p times.

Lemma 6. *Any $(2p+1)$ -dimensional 2-step nilpotent Lie algebra with characteristic sequence $(2, \dots, 2, 1)$ is isomorphic to a linear deformation of $\mathfrak{g}_{p,1}$.*

Proof. Let \mathfrak{g} be a $(2p+1)$ -dimensional 2-step nilpotent Lie algebra with characteristic sequence $(2, \dots, 2, 1)$. There exists a basis $\{X_1, \dots, X_{2p+1}\}$ such that the characteristic sequence is given by the operator $ad(X_1)$. If $\{X_1, \dots, X_{2p+1}\}$ is the Jordan basis of $ad(X_1)$ then the brackets of \mathfrak{g} write

$$\left\{ \begin{array}{l} [X_1, X_{2i}] = X_{2i+1}, \quad 1 \leq i \leq p, \\ [X_{2i}, X_{2j}] = \sum_{k=1}^p a_{2i, 2j}^{2k+1} X_{2k+1}, \quad 1 \leq i < j \leq p. \end{array} \right.$$

The change of basis $Y_1 = X_1$, $Y_i = tX_i$ for $2 \leq i \leq 2p+1$ shows that \mathfrak{g} is isomorphic to \mathfrak{g}_t whose brackets is $\mu_t = \mu_0 + t\varphi$ where μ_0 the multiplication of $\mathfrak{g}_{p,1}$ and $\varphi(X_{2i}, X_{2j}) = \sum_{k=1}^p a_{2i, 2j}^{2k+1} X_{2k+1}$, $1 \leq i < j \leq p$, $\varphi(X_l, X_s) = 0$ for all the other cases with $l < s$. So \mathfrak{g} is a linear deformation of $\mathfrak{g}_{p,1}$. Let us compute the second cohomological group $H_{CH}^2(\mathfrak{g}_{p,1}, \mathfrak{g}_{p,1})$. It is clear that every 2-cocycle φ is cohomologous to a cocycle φ' with $\varphi'(X_1, X_{2i}) = 0$ for

$1 \leq i \leq p$. Then we can assume that φ satisfies these relations : $\varphi(X_1, Y) = 0$ for any Y in $\mathfrak{g}_{p,1}$. In this case, we have

$$0 = \delta_{CH, \mu_0} \varphi(X_1, X_{2i}, X_l) = \varphi(X_{2i+1}, X_l),$$

for any $1 \leq l \leq 2p+1$ and $1 \leq i \leq p$. If we put $\varphi(X_{2i}, X_{2j}) = \sum_{k=1}^{2p+1} a_{2i,2j}^k X_k$, the equations $\delta_{CH, \mu_0} \varphi(X_{2i}, X_{2j}, X_l) = 0$ for $l = 1, 2$ implies that $a_{2i,2j}^{2k} = a_{2i,2j}^1 = 0$ for $1 \leq k \leq p$. But $\delta_{CH, \mu_0} f(X_{2i}, X_{2j}) = a_{1,2i} X_{2j+1} - a_{1,2j} X_{2i+1}$ and φ is cohomologous to

$$(4) \quad \left\{ \begin{array}{l} \varphi(X_2, X_4) = \sum_{k=3}^p a_{24}^{2k+1} X_{2k+1}, \\ \varphi(X_2, X_{2i}) = \sum_{k=2}^p a_{2,2i}^{2k+1} X_{2k+1}, \quad 3 \leq i \leq p, \\ \varphi(X_{2i}, X_{2j}) = \sum_{k=1}^p a_{2i,2j}^{2k+1} X_{2k+1} \quad 2 \leq i < j \leq p, \end{array} \right.$$

if $p \geq 3$. If $p = 2$ the cocycle φ is trivial. We deduce that for any 2-step nilpotent $(2p+1)$ -dimensional Lie algebra \mathfrak{g} with characteristic sequence $(2, \dots, 2, 1)$, there exists $\varphi \in Z_{CH}^2(\mathfrak{g}_{p,1}, \mathfrak{g}_{p,1})$ such that \mathfrak{g} is isomorphic to the following Lie algebra:

$$(5) \quad \left\{ \begin{array}{l} [X_1, X_{2i}] = X_{2i+1}, \quad 1 \leq i \leq p, \\ [X_{2i}, X_{2j}] = \varphi(X_{2i}, X_{2j}), \quad 2 \leq i < j \leq p. \end{array} \right.$$

where φ satisfies (4). In fact, if φ satisfies (4), it also satisfies $\varphi \bullet \varphi = 0$. In particular the subspace \mathfrak{m} generated by $\{X_2, \dots, X_{2p+1}\}$ is a 2-step nilpotent Lie subalgebra of \mathfrak{g} . For example, if $p = 3$, \mathfrak{m} is a 6-dimensional 2-step nilpotent Lie algebra isomorphic to one of the following:

- $[X_2, X_4] = X_7, [X_2, X_6] = X_5, [X_4, X_6] = X_3$, if $c(\mathfrak{m}) = (2, 2, 1, 1)$,
- $[X_2, X_6] = X_5, [X_4, X_6] = X_3$, if $c(\mathfrak{m}) = (2, 2, 1, 1)$,
- $[X_2, X_4] = X_7$ if $c(\mathfrak{m}) = (2, 1, 1, 1, 1)$,
- $[X_2, X_6] = X_7$ if $c(\mathfrak{m}) = (2, 1, 1, 1, 1)$,
- $[X_2, X_i] = 0$ if \mathfrak{m} is abelian.

We find again the classification of 7-dimensional 2-step nilpotent Lie algebras with characteristic sequence $(2, 2, 2, 1)$.

Let $\mathcal{F}_{k_2=p, k_1=1}^{2p+1,2} = \mathcal{F}_{p,1}^{2p+1,2}$ be the family of 2-step nilpotent $(2p+1)$ -dimensional Lie algebras of characteristic sequence $(2, \dots, 2, 1)$. It is the orbit associated with the action of the algebraic group $GL(2p+1, \mathbb{K})$ of the family $\mathfrak{f}_{p,1}^{2p+1,2}$ of Lie algebras given in (5). Since any φ of the family (5) is given by (4) and satisfies $\varphi \bullet \varphi = 0$, the family $\mathfrak{f}_{p,1}^{2p+1,2}$ is parametrized by the structure constants (a_{ij}^k) of φ and $\mathfrak{f}_{p,1}^{2p+1,2}$ is a linear plane in \mathbb{K}^{n^3} . Then $\mathcal{F}_{p,1}^{2p+1,2}$ is a connected algebraic variety which is fibered by the orbits of the Lie algebras of $\mathfrak{f}_{p,1}^{2p+1,2}$. In particular, if there exists in $\mathcal{F}_{p,1}^{2p+1,2}$ a rigid Lie algebra in $2Nilp_{2p+1}$, that is, a Lie algebra of $\mathcal{F}_{p,1}^{2p+1,2}$ whose orbit is Zariski-open, its Zariski closure is an algebraic component of $\mathcal{F}_{p,1}^{2p+1,2}$ and coincides with it. This implies

Proposition 7. *If \mathfrak{g} and \mathfrak{g}' are two $(2p+1)$ -dimensional Lie algebras rigid in $2Nilp_{2p+1}$ and belonging to $\mathcal{F}_{p,1}^{2p+1,2}$ then they are isomorphic.*

Let us determine these Lie algebras.

Proposition 8. (1) *The Lie algebra $\mathfrak{g}_5 = \mathfrak{g}_{2,1}$ is rigid in $2Nilp_5$.*

(2) *Let \mathfrak{g}_7 be the 7-dimensional 2-step nilpotent Lie algebra defined by*

$$[X_1, X_{2i}] = X_{2i+1}, 1 \leq i \leq 3, [X_2, X_4] = X_7, [X_2, X_6] = X_5, [X_4, X_6] = X_3.$$

This Lie algebra is rigid in $2Nilp_7$ and any Lie algebra belonging to $\mathcal{F}_{3,1}^{7,2}$ is isomorphic to a contraction of \mathfrak{g}_7 .

(3) *Let \mathfrak{g}_9 be the 9-dimensional Lie algebra given by*

$$[X_1, X_{2i}] = X_{2i+1}, 1 \leq i \leq 4, [X_2, X_4] = X_7, [X_2, X_8] = X_5, [X_4, X_6] = X_9, [X_6, X_8] = X_3.$$

This Lie algebra is rigid in $2Nilp_9$ and any Lie algebra belonging to $\mathcal{F}_{4,1}^{9,2}$ is isomorphic to a contraction of \mathfrak{g}_9 .

Proof. In fact, the linear space of 2-cocycles $\varphi \in Z_{CH}^2(\mathfrak{g}, \mathfrak{g})$ satisfying Equations (4) is of dimension $m_p = \frac{p(p+1)(p-2)}{2}$. For $p = 2$, we obtain $m_2 = 0$ and $\mathfrak{g}_{2,1}$ is rigid in $2Nilp_5$. For $p = 3$ and 4, we have respectively $m_3 = 6$ and $m_4 = 20$. This corresponds to the dimension of the 2-coboundaries satisfying $\delta_{CH,\mu}f(X_1, X_i) = 0$. We deduce that, in each case, $\dim H_{CH}^2(\mathfrak{g}, \mathfrak{g}) = 0$ and \mathfrak{g}_7 and \mathfrak{g}_9 are rigid.

Remark. Since $\mathcal{F}_{3,1}^{7,2}$ and $\mathcal{F}_{4,1}^{9,2}$ are connected, they are respectively the closure of the orbit of \mathfrak{g}_7 and \mathfrak{g}_9 . Then any rigid Lie algebra in $\mathcal{F}_{3,1}^{7,2}$ (respectively $\mathcal{F}_{4,1}^{9,2}$) is isomorphic to \mathfrak{g}_7 (respectively \mathfrak{g}_9). Others Lie algebras of these families are in the boundary of the closure of the orbit of \mathfrak{g}_7 or \mathfrak{g}_9 and are, by definition, contractions of \mathfrak{g}_7 or \mathfrak{g}_9 . (see [4] for definition of contractions).

Proposition 9. *If $p \geq 5$, any 2-step nilpotent of dimension $2p+1$ with characteristic sequence $(2, \dots, 2, 1)$ is not rigid in $2Nilp_{2p+1}$.*

Proof (1) We first remark that a Lie algebra of $\mathcal{F}_{p,1}^{2p+1,2}$ is rigid in $2Nilp_{2p+1}$ if and only if its second space of cohomology H_{CH}^2 is trivial. In fact, Nijenhuis-Richardson Theorem implies that a Lie algebra in $2Nilp_{2p+1}$ with a trivial second cohomology space is rigid in $2Nilp_{2p+1}$. For the converse we know that if a rigid Lie algebra has a nontrivial cohomology, its affine scheme associated with the algebraic variety $2Nilp_{2p+1}$ is not reduced. But the family $\mathcal{F}_{p,1}^{2p+1,2}$ is isomorphic to a plane and its associated affine scheme is reduced. Then here the converse of Nijenhuis-Richardson Theorem is true and we have to prove that $\dim H_{CH}^2(\mathfrak{g}, \mathfrak{g}) \neq 0$. To compute $Z_{CH}^2(\mathfrak{g}, \mathfrak{g})$, we can consider only the 2-cocycles satisfying $\varphi(X_1, Y) = 0$ for any $Y \in \mathfrak{g}$. This implies that φ is completely determinate by its constants

$$\varphi(X_{2i}, X_{2j}) = \sum a_{ij}^k X_{2k+1}.$$

Then this space is of dimension $\frac{p^2(p-1)}{2}$. Consider now a linear endomorphism of \mathfrak{g} such that $\delta_{CH,\mu}f(X_1, X_i) = 0$. This conditions implies that $f(X_{2i+1}) = [f(X_1), X_{2i}] + [X_1, f(X_{2i})]$. If we write

$$\delta_{CH,\mu}f(X_{2i}, X_{2j}) = \sum b_{ij}^k X_{2k+1}$$

the coefficients b_{ij}^k are linear combinations of the coefficients $\alpha_{1,1}, \alpha_{1,2i}, i = 1, \dots, p$ and $\alpha_{2i,1}, \alpha_{2i,2j}$ for $i, j = 1, \dots, p$ where the coefficients $\alpha_{i,j}$ are the coefficients of the matrix of f in the basis $\{X_1, \dots, X_{2p+1}\}$. Then the b_{ij}^k are linear combinations of $(p+1)^2$ parameters. But we have $\frac{p^2(p-1)}{2}$ coefficients a_{ij}^k . Since

$$\frac{p^2(p-1)}{2} \leq (p+1)^2 \Rightarrow p < 5$$

then $\dim B_{CH}^2(\mathfrak{g}, \mathfrak{g}) < \dim Z_{CH}^2(\mathfrak{g}, \mathfrak{g})$ as soon as $p \geq 5$.

As soon as $p > 5$ there is no rigid Lie algebras in $2Nilp_n$. A family with k parameters of Lie algebras of $2Nilp_n$ is called irreducible if two algebras corresponding to different values of the parameters give two non isomorphic algebras. A irreducible family with k parameters is rigid if any deformation of an algebra of this family is isomorphic to an algebra of this family. This is equivalent to say that the orbit of this family is open. For $p > 5$ there exists a irreducible family with k parameters rigid in $2Nilp_n$ and any algebra in $2Nilp_n$ is a contraction of an element of this family.

2.5. 2-step nilpotent Lie algebras with characteristic sequence $(2, \dots, 2, 1, 1)$. Let us denote by $\mathfrak{g}_{p-1,2}$ the $(2p)$ -dimensional Lie algebra given by the brackets

$$[X_1, X_{2i}] = X_{2i+1}, \quad 1 \leq i \leq p-1,$$

other non defined brackets are equal to zero.

Lemma 10. *Any 2-step nilpotent $(2p)$ -dimensional Lie algebra with characteristic sequence $(2, \dots, 2, 1, 1)$ is isomorphic to a linear deformation of $\mathfrak{g}_{p-1,2}$. Its Lie bracket is isomorphic to $\mu = \mu_0 + t\varphi$, where μ_0 is the Lie bracket of $\mathfrak{g}_{p-1,2}$, and φ is a skew-bilinear form such that*

$$(6) \quad \begin{cases} \varphi \in Z_{CH}^2(\mathfrak{g}_{p-1,2}, \mathfrak{g}_{p-1,2}), \\ \varphi(\varphi(X, Y), Z) = 0, \quad \forall X, Y, Z. \end{cases}$$

Proof. The proof is similar to the proof of Lemma 6. Note that, for a general linear deformation $\mu = \mu_0 + t\varphi$, the map φ is a 2-cocycle for the Chevalley-Eilenberg cohomology of μ_0 satisfying also the Jacobi condition ([16]). But, since μ is 2-step nilpotent, this condition reduces to $\varphi(\varphi(X, Y), Z) = 0$ for any X, Y, Z .

Lemma 11. *Any cocycle in $Z_{CH}^2(\mathfrak{g}_{p-1,2}, \mathfrak{g}_{p-1,2})$ is cohomologous to a cocycle satisfying*

$$(7) \quad \begin{cases} \varphi(X_1, X_i) = 0, i = 1, \dots, 2p-1, \quad \varphi(X_1, X_{2p}) = aX_{2p}, \\ \varphi(X_{2j+1}, X_k) = 0, i = 1, \dots, p, \quad j = 1, \dots, p-1, \quad k = 1, \dots, 2p, \\ \varphi(X_{2i}, X_{2j}) = \sum_{k=1}^p a_{2i,2j}^{2k+1} X_{2k+1} + a_{2i,2j}^{2p} X_{2p}, \quad 2 \leq i < j \leq p. \end{cases}$$

The proof results of a direct computation. The coboundaries of Type (7) satisfy

$$\begin{cases} \delta f(X_{2i}, X_{2j}) = a_{1,2i} X_{2j+1} - a_{1,2j} X_{2i+1}, \quad 1 \leq i < j \leq p-1, \\ \delta f(X_{2i}, X_{2p}) = -a_{1,2p} X_{2i+1}, \quad 1 \leq i \leq p-1. \end{cases}$$

We deduce

$$(8) \quad \dim H_{CH}^2(\mathfrak{g}_{p-1,2}, \mathfrak{g}_{p-1,2}) = \frac{p^3 - p^2 - 2p + 2}{2}.$$

For example, $\dim H_{CH}^2(\mathfrak{g}_{p-1,2}, \mathfrak{g}_{p-1,2}) = 0$ if and only if $p = 1$ and in this case $\mathfrak{g}_{0,2}$ is the 2-dimensional abelian Lie algebra and we know that this Lie algebra is rigid in the variety of nilpotent Lie algebra of dimension 2. If $p > 1$, then $H_{CH}^2(\mathfrak{g}_{p-1,2}, \mathfrak{g}_{p-1,2})$ is not trivial.

Let $\mathcal{F}_{k_2=p-1, k_1=2}^{2p,2} = \mathcal{F}_{p-1,2}^{2p,2}$ be the family of 2-step nilpotent $(2p)$ -dimensional Lie algebra with characteristic sequence $(2, \dots, 2, 1, 1)$. Any element of this family is isomorphic to a linear deformation of $\mathfrak{g}_{p-1,2}$ whose bracket satisfy $\mu = \mu_0 + \varphi$ with φ satisfies (6). Let $\varphi \in Z_{CH}^2(\mathfrak{g}_{p-1,2}, \mathfrak{g}_{p-1,2})$ given by (7). The condition $\varphi(\varphi(X, Y), Z) = 0$ is equivalent to:

- (1) $a = 0, a_{2i,2p}^{2p} = 0, 1 \leq i \leq p-1$
- (2) and one of the following conditions:
 - (a) $a_{2i,2j}^{2p} = 0, 1 \leq i < j \leq p-1$
 - (b) $\varphi(X_{2i}, X_{2p}) = 0, 1 \leq i \leq p-1$

Proposition 12. *The family $\mathcal{F}_{p-1,2}^{2p,2}$ of 2-step nilpotent $(2p)$ -dimensional Lie algebras with characteristic sequence $(2, \dots, 2, 1, 1)$ is the union of two algebraic components, the first one, $\mathcal{C}_1(\mathcal{F}_{p-1,2}^{2p,2})$, corresponds to the cocycles*

$$(9) \quad \varphi(X_{2i}, X_{2j}) = \sum_{k=1}^{p-1} a_{2i,2j}^{2k+1} X_{2k+1}, \quad 2 \leq i < j \leq p,$$

the second one, $\mathcal{C}_2(\mathcal{F}_{p-1,2}^{2p,2})$, to the cocycles

$$(10) \quad \varphi(X_{2i}, X_{2j}) = \sum_{k=1}^p a_{2i,2j}^{2k+1} X_{2k+1} + a_{2i,2j}^{2p} X_{2p}, \quad 2 \leq i < j \leq p-1,$$

where the non defined product $\varphi(X, Y)$ are nul.

Each one of these components is a regular algebraic variety. These components can be characterized by a property of the center: it is generated by $\{X_{2i+1}, i = 1, \dots, p-1\}$ for $\mathcal{C}_1(\mathcal{F}_{p-1,2}^{2p,2})$ and by $\{X_{2i+1}, i = 1, \dots, p-1, X_{2p}\}$ for $\mathcal{C}_2(\mathcal{F}_{p-1,2}^{2p,2})$.

Proposition 13. (1) *Assume that $p \geq 5$. A $2p$ -dimensional Lie algebra of $\mathcal{C}_1(\mathcal{F}_{p-1,2}^{2p,2})$ is never rigid.*

(2) *Let $p = 3$ and \mathfrak{g}_6 be the Lie algebra of $\mathcal{C}_1(\mathcal{F}_{2,2}^{6,2})$ given by*

$$[X_1, X_{2i}] = X_{2i+1}, i = 1, 2, \quad [X_2, X_6] = X_5, \quad [X_4, X_6] = X_3.$$

Then \mathfrak{g}_6 is rigid in 2Nilp_6 .

(3) *Let $p = 4$ and \mathfrak{g}_8 be the Lie algebra of $\mathcal{C}_1(\mathcal{F}_{3,2}^{8,2})$ given by*

$$[X_1, X_{2i}] = X_{2i+1}, i = 1, 2, 3, \quad [X_2, X_4] = X_7, \quad [X_4, X_8] = X_3, \quad [X_6, X_8] = X_5.$$

Then \mathfrak{g}_8 is rigid in 2Nilp_8 .

Proof.

- (1) Let \mathfrak{g} be in $\mathcal{C}_1(\mathcal{F}_{p-1,2}^{2p,2})$. A 2-cocycle belonging to $Z_{CH}^2(\mathfrak{g}, \mathfrak{g})$ is cohomologous to a cocycle of type (9). The space of such cocycles is of dimension $\frac{p(p-1)^2}{2}$. The coboundary belonging to this space are parametrized by the coefficients $a_{11}, a_{1,i}$ and $a_{i,1}$ for $i = 2, \dots, 2p$, $a_{2k,2l}$ for $k, l = 1, \dots, p$, that is, $(p+1)^2$ coefficients. As soon as $p \geq 5$ we have $\frac{p(p-1)^2}{2} > (p+1)^2$. Moreover any rigid Lie algebra in $\mathcal{C}_1(\mathcal{F}_{p-1,2}^{2p,2})$ has a second group of cohomology of dimension 0. So there are no rigid Lie algebras in $\mathcal{C}_1(\mathcal{F}_{p-1,2}^{2p,2})$ for $p \geq 5$.
- (2) and (3) The Lie algebras \mathfrak{g}_6 and \mathfrak{g}_8 satisfy $\dim H_{CH}^2(\mathfrak{g}, \mathfrak{g}) = 0$. Thus they are rigid. Remark that the algebra \mathfrak{g}_8 corresponds to the algebra $N_9^{8,3}$ in the classification of [18]. The algebra $N_5^{8,3}, N_6^{8,3}, N_7^{8,3}, N_8^{8,3}$ which are the other algebras belonging to $\mathcal{C}_1(\mathcal{F}_{3,2}^{8,2})$ are contractions of the algebra $N_9^{8,3}$.

Proposition 14. (1) Assume that $p \geq 6$. A $2p$ -dimensional Lie algebra of $\mathcal{C}_2(\mathcal{F}_{p-1,2}^{2p,2})$ is never rigid.

- (2) Let $p = 3$ and \mathfrak{h}_6 be the Lie algebra of $\mathcal{C}_2(\mathcal{F}_{2,2}^{6,2})$ given by

$$[X_1, X_{2i}] = X_{2i+1}, i = 1, 2, [X_2, X_4] = X_6.$$

Then \mathfrak{h}_6 is rigid in $2Nilp_6$.

- (3) Let $p = 4$ and \mathfrak{h}_8 be the Lie algebra of $\mathcal{C}_2(\mathcal{F}_{3,2}^{8,2})$ given by

$$[X_1, X_{2i}] = X_{2i+1}, i = 1, 2, 3, [X_2, X_6] = X_5, [X_2, X_4] = X_8.$$

Then \mathfrak{h}_8 is rigid in $2Nilp_8$.

- (4) Let $p = 5$ and \mathfrak{h}_{10} be the Lie algebra of $\mathcal{C}_2(\mathcal{F}_{4,2}^{10,2})$ given by

$$[X_1, X_{2i}] = X_{2i+1}, i = 1, 2, 3, 4, [X_2, X_4] = X_{10}, [X_2, X_6] = X_5, [X_2, X_8] = X_3,$$

$$[X_4, X_6] = X_9, [X_4, X_8] = X_7, [X_6, X_8] = X_3.$$

Then \mathfrak{h}_{10} is rigid in $2Nilp_{10}$.

The proof of the proposition is analogous to the proof of Proposition 13.

3. CUBIC ASSOCIATIVE LIE MULTIPLICATION

3.1. The variety $3Nilp_n$ of n -dimensional 3-step nilpotent Lie algebras and cubic associative algebras. Let A be a \mathbb{K} -associative algebra with binary multiplication xy . The associativity which is the quadratic relation

$$(xy)z = x(yz)$$

implies six cubic relations

$$(11) \quad \begin{cases} ((xy)z)t = (x(yz))t, \\ (x(yz))t = x((yz)t), \\ x((yz)t) = x(y(zt)), \\ x(y(zt)) = (xy)(zt), \\ (xy)(zt) = ((xy)z)t. \end{cases}$$

Recall that these relations correspond to the edges of the Stasheff pentagon.

Definition 15. A binary algebra, that is, an algebra whose multiplication is given by a bilinear map, is called cubic associative if the multiplication satisfies the cubic relations (11).

We call these relations cubic because if we denote by μ the multiplication, it occurs exactly three times in each term of the relations. For example, the first relation writes as

$$\mu \circ (\mu \circ (\mu \otimes Id) \otimes Id) = \mu \circ (\mu \circ (Id \otimes \mu) \otimes Id)$$

which is cubic in μ . It is the same thing for all other relations.

Proposition 16. *Let \mathfrak{g} be a Lie algebra. The Lie bracket is cubic associative if and only if \mathfrak{g} is 3-step nilpotent.*

In fact, the first identity of (11) becomes

$$\begin{aligned} [[[X_1, X_2], X_3], X_4] &= [[X_1, [X_2, X_3]], X_4] \\ &= -[[[X_2, X_3]], X_1], X_4] \end{aligned}$$

and finally

$$[[[X_1, X_2], X_3], X_4] + [[[X_2, X_3]], X_1], X_4] = -[[[X_3, X_1]], X_2], X_4] = 0,$$

which implies that \mathfrak{g} is 3-nilpotent. Conversely, if \mathfrak{g} is 3-nilpotent, all the relations of (11) are satisfied.

Let $\mathfrak{g} = (\mathbb{K}^n, \mu)$ be a 3-step nilpotent n -dimensional Lie algebra and $\{X_1, \dots, X_n\}$ be a basis of \mathbb{K}^n . The structure constants (C_{ij}^k) of μ related to this basis satisfy the following polynomial equations:

$$(12) \quad \left\{ \begin{array}{l} C_{ij}^k = -C_{ji}^k, \\ \sum_{l=1}^n C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{lj}^s = 0, \quad s = 1, \dots, n, \\ \sum_{t,u=1}^n C_{ij}^t C_{tk}^u C_{ul}^s = 0, \quad s = 1, \dots, n \end{array} \right.$$

for any $i, j, k, l \in \{1, \dots, n\}$. Let \mathfrak{g} be a n -dimensional 3-step nilpotent Lie algebra. If we consider the vector space \mathbb{K}^{n^3} parametrized by the structure constants C_{ij}^k , the polynomial equations (12) are the equations of an algebraic variety in \mathbb{K}^{n^3} . We denote this variety by $3Nilp_n$. Any element of $3Nilp_n$ is a Lie bracket μ of a 3-step nilpotent, n -dimensional Lie algebra $\mathfrak{g} = (\mathbb{K}^n, \mu)$. We identify \mathfrak{g} with its Lie bracket μ .

Let $\mathfrak{g} \in 3Nilp_n$. Its characteristic sequence is of type $(3, \dots, 3, 2, \dots, 2, 1, \dots, 1)$ with $n = 3k_3 + 2k_2 + k_1$ where k_i is the number of i in the characteristic sequence. The aim of the next section is to describe the families $\mathcal{F}_{k_3, k_2, k_1}^{n, 3}$ of these Lie algebras. For example, in dimension 7, these families are $\mathcal{F}_{2,0,1}^{7,3}$, $\mathcal{F}_{1,1,2}^{7,3}$ and $\mathcal{F}_{1,0,4}^{7,3}$.

3.2. Deformations and rigidity in $3Nilp_n$. Let $\mathfrak{g}_{k_3, k_2, k_1}$ be the $n = 3k_3 + 2k_2 + k_1$ -dimensional nilpotent Lie algebra with characteristic sequence $(3, \dots, 3, 2, \dots, 2, 1, \dots, 1)$ belonging to $\mathcal{F}_{k_3, k_2, k_1}^{n, 3}$ given by

$$(13) \quad \left\{ \begin{array}{l} [X_1, X_{2+3i}] = X_{3+3i}, \quad [X_1, X_{3+3i}] = X_{4+3i}, \quad i = 0, \dots, k_3 - 1, \\ [X_1, X_{3k_3+2j}] = X_{3k_3+2j+1}, \quad j = 1, \dots, k_2, \\ [X_1, X_{3k_1+2k_2+l+1}] = 0, \quad l = 1, \dots, k_1 - 1. \end{array} \right.$$

Lemma 17. *Any $n = 3k_3 + 2k_2 + k_1$ -dimensional 3-step nilpotent Lie algebra \mathfrak{g} , i.e $\mathfrak{g} \in \mathcal{F}_{k_3, k_2, k_1}^{n, 3}$, is a linear deformation of $\mathfrak{g}_{k_3, k_2, k_1}$.*

The proof is similar to the proof of Lemma 6.

As consequence, if μ_0 is the Lie bracket of $\mathfrak{g}_{k_3, k_2, k_1}$, the Lie bracket μ of \mathfrak{g} is isomorphic to $\mu_0 + t\varphi$ where φ is a skew-bilinear map satisfying:

$$(14) \quad \begin{cases} \delta_{C, \mu_0}^2(\varphi) = 0, \\ \varphi \bullet \varphi = 0, \\ \delta_{R, \mu_0}^2(\varphi) = 0, \\ \mu_0 \circ_1 \varphi \circ_1 \varphi + \varphi \circ_1 \varphi \circ_1 \mu_0 + \varphi \circ_1 \mu_0 \circ_1 \varphi = 0, \\ \varphi \circ_1 \varphi \circ_1 \varphi = 0, \end{cases}$$

where $\delta_{R, \mu_0}^2(\varphi)$ is the 4-linear map

$$\delta_{R, \mu_0}^2(\varphi) = \mu_0 \circ_1 \mu_0 \circ_1 \varphi + \mu_0 \circ_1 \varphi \circ_1 \mu_0 + \varphi \circ_1 \mu_0 \circ_1 \mu_0,$$

the identity

$$\varphi \bullet \varphi = 0$$

is the Jacobi identity, and \circ_1 is the comp_1 -operation (composition on the first argument). If we put $\delta_{R, \mu}^1(f) = \delta_{C, \mu}^1(f)$ for any linear endomorphism f on a Lie algebra then

$$\delta_{R, \mu_0}^2(\delta_{R, \mu_0}^1(f)) = 0.$$

Let $\mathfrak{g} = (\mathbb{K}^n, \mu)$ be a n -dimensional \mathbb{K} -Lie algebra. We consider the complex

$$(\mathcal{D}^i(\mathbb{K}^n, \mathbb{K}^n), \delta_{CR, \mu}^i)_{i \geq 1}$$

where the space $\mathcal{D}^i(\mathbb{K}^n, \mathbb{K}^n)$ of i -cochains is $\mathcal{D}^i(\mathbb{K}^n, \mathbb{K}^n) = \mathcal{C}^i(\mathbb{K}^n, \mathbb{K}^n)$ for $i = 1, 2$ and $\mathcal{C}^i(\mathbb{K}^n, \mathbb{K}^n) \oplus \mathcal{C}^i(\mathbb{K}^n, \mathbb{K}^n)$ for $i \geq 3$. The coboundary operators are given by $\delta_{CR, \mu}^1 = \delta_{C, \mu}^1$, $\delta_{CR, \mu}^2(\varphi) = (\delta_{C, \mu}^2(\varphi), \delta_{R, \mu}^2(\varphi))$, $\delta_{CR}^3(\phi, \psi) = (\delta_{C, \mu}^3(\phi), \delta_{R, \mu}^3(\psi))$ and

$$\delta_{R, \mu}^3(\psi)(X_1, \dots, X_5) = \mu(\psi(X_1, \dots, X_4), X_5) - \psi(\mu(X_1, X_2), X_3, X_4, X_5).$$

The associated cohomology is the deformation cohomology for the 3-step nilpotent Lie algebras. To illustrate, we compute some spaces of this cohomology for particular 3-step nilpotent Lie algebras and describe the link with the problem of classification.

Examples

- (1) *Characteristic sequence (3, 1, 1).* Let $\mathfrak{g}_{1,0,2}$ be the 5-dimensional Lie algebra defined by

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4.$$

We denote by μ_0 its Lie bracket. Let $\mu_0 + t\varphi$ be a linear deformation of μ_0 . The bilinear map φ satisfies relation (14). It is cohomologous to a cocycle, still denoted by φ , which satisfies $\varphi(X_1, Y) = 0$ for any Y . This implies that $Z_{CR}^2(\mu_0, \mu_0)$ is constituted of bilinear forms φ defined by

$$\varphi(X_2, X_3) = aX_5, \quad \varphi(X_2, X_5) = bX_4 + cX_5.$$

The quadratic condition $\mu_0 \circ \varphi \circ \varphi + \varphi \circ \varphi \circ \mu_0 + \varphi \circ \mu_0 \circ \varphi = 0$ implies that $ab = 0$ and the ternary condition $\varphi \circ \varphi \circ \varphi = 0$ gives $c = 0$. We deduce that φ satisfies

$$\varphi(X_2, X_3) = aX_5, \quad \varphi(X_2, X_5) = bX_4$$

with $ab = 0$. We find the classification of 5-dimensional 3-step nilpotent Lie algebras again.

- (2) *Characteristic sequence* $(3, 1, \dots, 1)$ and $n = 3 + p$. If \mathfrak{g} is such a Lie algebra, its Lie bracket μ is isomorphic to a linear deformation of μ_0 , the Lie bracket of $\mathfrak{g}_{1,0,p}$. The bilinear form φ satisfies conditions (14), and we can assume that $\varphi(X_1, Y) = 0$ for any Y . This implies in particular that $\varphi(X_4, Y) = 0$. We consider the 2-cocycles of $Z_{CR}^2(\mu_0, \mu_0)$ satisfying these conditions. They are cohomologous to 2-cocycles given by

$$\begin{cases} \varphi(X_2, X_3) = \sum_{i \geq 5} a_{2,3}^i X_i, \\ \varphi(X_2, X_k) = \sum_{i \geq 4} a_{2,k}^i X_i, \quad 5 \leq k \leq n, \\ \varphi(X_l, X_k) = \sum_{i \geq 4} a_{l,k}^i X_i, \quad 5 \leq l < k \leq n. \end{cases}$$

In particular $\dim H_{CR}^2(\mu_0, \mu_0) = (n - 3) \frac{n^2 - 7n + 14}{2} - 1$. Since the characteristic sequence is equal to $(3, 1, \dots, 1)$ we have necessarily $\varphi(X_l, X_k) = a_{l,k}^4 X_4$ for $5 \leq l < k \leq n$ and also for $l = 2, k \geq 5$. If $\varphi(X_2, X_3)$ is not 0, then all the constants $a_{l,k}^4$ are 0 and \mathfrak{g} is a direct product of the 5-dimensional Lie algebra described above and an abelian ideal. If we assume that \mathfrak{g} is indecomposable, then $\varphi(X_2, X_3) = 0$ and \mathfrak{g} is isomorphic to the Lie algebra

$$(15) \quad \begin{cases} [X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \\ [X_2, X_k] = a_{2,k} X_4, \quad k \geq 5, \\ [X_l, X_k] = a_{l,k}^4 X_4, \quad 5 \leq l < k \leq n. \end{cases}$$

3.3. Cubic operads. If $\mathcal{A}_{ss} = \Gamma(E)/(R_{\mathcal{A}_{ss}})$ is the operad for associative algebras, the relations (11) are the generating relations of $(R_{\mathcal{A}_{ss}})(4)$. But these relations are following from the relations defining $(R_{\mathcal{A}_{ss}})(3) = R_{\mathcal{A}_{ss}}$. In Definition 15, we do not assume that the algebra is associative. It is clear that (11) do not implies associativity. From the relations (11) we can define a binary cubic operad $\mathcal{A}_{ss}Cubic$.

Let E be a $\mathbb{K}[\Sigma_2]$ -module and $\Gamma(E)$ the free operad generated by E . Consider a $\mathbb{K}[\Sigma_4]$ -submodule R of $\Gamma(E)(4)$. Let \mathcal{R} the ideal of $\Gamma(E)$ generated by R . We have

$$\mathcal{R} = \{\mathcal{R}(n), n \in \mathbb{N}^*\}$$

with $\mathcal{R}(1) = \{0\}$, $\mathcal{R}(2) = \{0\}$, $\mathcal{R}(3) = \{0\}$, $\mathcal{R}(4) = R$.

Definition 18. We call cubic operad generated by E and defined by the relations $R \subset \Gamma(E)(4)$, the operad $\mathcal{P}(E, R)$ given by

$$\mathcal{P}(E, R)(n) = \frac{\Gamma(E)(n)}{\mathcal{R}(n)}.$$

For example, the operad $\mathcal{A}_{ss}Cubic$ is the cubic operad generated by $E = \mathbb{K}[\Sigma_2]$ and the $\mathbb{K}[\Sigma_4]$ -submodule of relations R generated by the vectors

$$\begin{cases} ((x_1 x_2) x_3) x_4 - (x_1 (x_2 x_3)) x_4, (x_1 (x_2 x_3)) x_4 - x_1 ((x_2 x_3) x_4), x_1 ((x_2 x_3) x_4) - x_1 (x_2 (x_3 x_4)), \\ x_1 (x_2 (x_3 x_4)) - (x_1 x_2) (x_3 x_4), (x_1 x_2) (x_3 x_4) - ((x_1 x_2) x_3) x_4. \end{cases}$$

Thus we have $\mathcal{A}_{ss}Cubic(2) = \mathbb{K}[\Sigma_2]$, $\mathcal{A}_{ss}Cubic(3) = \mathbb{K}[\Sigma_3]$, and $\mathcal{A}_{ss}Cubic(4) = \frac{\Gamma(E)(4)}{\mathcal{R}(4)}$ is the 24-dimensional \mathbb{K} -vector space generated by $\{((x_{\sigma(1)} x_{\sigma(2)}) x_{\sigma(3)}) x_{\sigma(4)}, \sigma \in \Sigma_4\}$.

In particular any $3\mathcal{N}ilp$ -algebra is a 3-step nilpotent Lie algebra. But this operad is cubic so we don't know direct links between an operadic cohomology and the cohomology of deformations.

$$(16) \quad \left\{ [X_1, X_{3i-1}] = X_{3i}, [X_1, X_{3i}] = X_{3i+1}, i = 1, \dots, p. \right.$$
$$(17) \quad \left\{ \begin{array}{l} \varphi(X_{3i-1}, X_{3j}) = \sum_{k=1}^p a_{3i-1, 3j}^k X_{3k+1}, \quad 1 \leq i \leq j \leq p, \\ \varphi(X_{3i}, X_{3j-1}) = \sum_{k=1}^p a_{3i, 3j-1} X_{3k+1}, \quad 1 \leq i < j \leq p, \\ \varphi(X_{3i-1}, X_{3j-1}) = \sum_{k=1}^p (a_{3i, 3j-1} + a_{3i-1, 3j}) X_{3k} + \sum_{k=1}^p a_{3i-1, 3j-1} X_{3k+1}, \\ \hspace{25em} 1 \leq i < j \leq p. \end{array} \right.$$
$$\dim H_{CR}^2(\mathfrak{g}_{p,0,1}, \mathfrak{g}_{p,0,1}) = \frac{p^2(3p-1)}{2}.$$
$$\begin{cases} f(X_{3i}) = [f(X_1), X_{3i-1}] + (X_1, f(X_{3i-1})), \\ f(X_{3i+1}) = [f(X_1), X_{3i}] + (X_1, f(X_{3i})). \end{cases}$$
$$\dim B_{CR}^2(\mathfrak{g}, \mathfrak{g}) \leq (3p+1)(p+1)$$

Proposition 20. *If $p \geq 4$, any $\mathfrak{g} \in \mathcal{F}_{p,0,1}^{3p+1,3}$ is not rigid.*

In particular, since $\mathcal{F}_{p,0,1}^{3p+1,3}$ is isomorphic to a linear space of dimension $\frac{p^2(3p-1)}{2}$, this family is the orbit of a family with at least $\frac{3p^3-7p^2-8p-2}{2}$ parameters.

Let us consider the case $p = 2$ that is of the dimension 7. We have

$$\begin{cases} \varphi(X_2, X_3) = a_1X_4 + b_1X_7, \\ \varphi(X_3, X_5) = a_2X_4 + b_2X_7, \\ \varphi(X_2, X_6) = a_3X_4 + b_3X_7, \\ \varphi(X_5, X_6) = a_4X_4 + b_4X_7, \\ \varphi(X_2, X_5) = (a_2 + a_3)X_3 + a_5X_4 + (b_2 + b_3)X_6 + b_5X_7. \end{cases}$$

In particular, computing $\delta_{C, \mu_0} f(X_i, X_j)$ with the condition $\varphi(X_1, Y) = 0$ for any Y , we find

$$\dim H^2(\mathfrak{g}_{2,0,1}, \mathfrak{g}_{2,0,1}) = 10$$

and the family $\mathcal{F}_{2,0,1}^{7,3}$ is reduced to a 10-dimensional regular algebraic variety parametrized by the constant (a_i, b_i) , $i = 1, \dots, 5$.

Proposition 21. *The 7-dimensional nilpotent Lie algebra \mathfrak{g} whose Lie bracket is*

$$(18) \quad \begin{cases} [X_1, X_{3i-1}] = X_{3i}, [X_1, X_{3i}] = X_{3i+1}, \quad i = 1, 2, \\ [X_2, X_3] = X_4, [X_3, X_5] = X_7, \\ [X_5, X_6] = X_4, \\ [X_2, X_5] = X_6. \end{cases}$$

is rigid in 3Nilp_7 . Any 7-dimensional Lie algebra of characteristic sequence $(3, 3, 1)$ is a contraction of \mathfrak{g} .

In fact $\dim H^2(\mathfrak{g}, \mathfrak{g}) = 0$.

We deduce that the classification of the Lie algebras of $\mathcal{F}_{2,0,1}^{7,3}$ is given by

$$\begin{aligned} (a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5) \in \{ & (0, 0, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 0, 0, 0, 1), \\ & (0, 0, 1, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 1, 0, 0, 0, 0, 0, 1), (0, 0, 0, 1, 0, 0, 0, 0, 0, 0), \\ & (0, 0, 0, 1, 0, 1, 0, 0, 0, 0), (1, 0, 0, 0, 0, 1, 0, 0, 1, 0), (1, 0, 0, 0, 0, 0, 0, 0, 1, 0), \\ & (1, 0, 0, 0, 1, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0, 0, 1, 0), (0, 0, 1, 1, 0, 0, 0, 0, 1, 0), \\ & (1, 0, 0, 1, 0, 0, 1, 0, 0, 0), (1, 0, 0, 1, 0, 0, 0, 0, 0, 1), (1, 0, 0, 0, 0, 0, 1, 0, 0, 0), \\ & (0, 0, 0, 0, 0, 0, 0, 0, 1, 0), (1, 0, 0, 0, 0, 0, 0, 0, 0, 1)\} \end{aligned}$$

3.5. Algebras attached to a 3-step nilpotent Lie algebra. If \mathfrak{g} is a 3-step nilpotent Lie algebra, there exists k_1, k_2, k_3 such that $\mathfrak{g} \in \mathcal{F}_{k_3, k_2, k_1}^{n, 3}$ and it is a linear deformation of $\mathfrak{g}_{k_3, k_2, k_1}$. Then the brackets of \mathfrak{g} and $\mathfrak{g}_{k_3, k_2, k_1}$ satisfy $\mu = \mu_0 + t\varphi$ and Relations (14). In particular we have

$$(19) \quad \mu_0 \circ_1 \varphi \circ_1 \varphi + \varphi \circ_1 \varphi \circ_1 \mu_0 + \varphi \circ_1 \mu_0 \circ_1 \varphi = 0.$$

This relation can be interpreted as a definition of an algebra structure on \mathbb{K}^n . This algebra is not necessarily a Lie algebra and it is associated with μ_0 .

Definition 22. *Let $\mathfrak{g} = (\mathbb{K}^n, \mu)$ be a n -dimensional 3-step nilpotent Lie algebra. We call algebra attached to \mathfrak{g} any n -dimensional \mathbb{K} -algebra whose bilinear multiplication φ satisfies the following identity:*

$$\mu \circ_1 \varphi \circ_1 \varphi + \varphi \circ_1 \varphi \circ_1 \mu + \varphi \circ_1 \mu \circ_1 \varphi = 0.$$

For example, if $\mathfrak{g} = \mathfrak{g}_{p,0,1} = (\mathbb{K}^{3p+1}, \mu_0)$, the algebras whose multiplication is (17) is attached to $\mathfrak{g}_{p,0,1}$.

For this type of algebras, we can define a cohomology of deformations which provides new invariants of \mathfrak{g} . For this cohomology, μ_0 is always a 2-cocycle.

Remark. There is a class of cubic algebras which is really interesting: the Jordan algebras. Recall that a \mathbb{K} -Jordan algebra is a commutative algebra satisfying the following identity

$$x(yx^2) = (xy)x^2.$$

Since \mathbb{K} is of zero characteristic, linearizing this identity, we obtain

$$((x_2x_3)x_4)x_1 + ((x_3x_1)x_4)x_2 + ((x_1x_2)x_4)x_3 - (x_2x_3)(x_4x_1) - (x_3x_1)(x_4x_2) - (x_1x_2)(x_4x_3) = 0.$$

This relation is cubic. It is invariant by the permutations $Id, \tau_{12}, \tau_{13}, \tau_{23}, c, c^2$ where c is the cycle (123). This permits to determinate an associated cubic operad called *Jord*. It is defined by

$$\mathcal{J}ord(2) = \mathbb{1}, \mathcal{J}ord(3) = \mathbb{1} \oplus V, \mathcal{J}ord(4) = \frac{\Gamma(\mathbb{1})(4)}{\mathcal{R}(4)}$$

where $\mathbb{1}$ is the identity representation of Σ_2 and the $\mathbb{K}[\Sigma_4]$ -module $\mathcal{R}(4)$ is generated by the vector

$$((x_2x_3)x_4)x_1 + ((x_3x_1)x_4)x_2 + ((x_1x_2)x_4)x_3 - (x_2x_3)(x_4x_1) - (x_3x_1)(x_4x_2) - (x_1x_2)(x_4x_3)$$

which is a vector space of dimension 4. Since $\dim \Gamma(\mathbb{1})(4) = 15$, thus $\frac{\Gamma(\mathbb{1})(4)}{\mathcal{R}(4)}$ is of dimension

11. But, as in the $\mathcal{N}ilp_3$ operad, we do not know a direct process giving an operadic cohomology or a link between this cohomology and the deformation cohomology. But we can directly define the space of bilinear forms associated with a linear deformation. This space corresponds to the 2-cocycles in the deformation cohomology. Using notations of [9], if μ is the Jordan multiplication, then the Jordan identity is written

$$(\mu \circ_1 \mu \circ_1 \mu - \mu \circ (\mu \otimes \mu)) \circ \Phi_v = 0$$

where v is the following vector of the group algebra $\mathbb{K}[\Sigma_4]$, $v = (2341) + (3142) + \tau_{34}$. We deduce that a 2-cocycle is defined by the identity

$$(\varphi \circ_1 \mu \circ_1 \mu + \mu \circ_1 \varphi \circ_1 \mu + \mu \circ_1 \mu \circ_1 \varphi - \mu \circ (\mu \otimes \varphi + \varphi \otimes \mu) + \varphi \circ (\mu \otimes \mu)) \circ \Phi_v = 0.$$

Remarks.

- (1) We can generalize this process to define $(n-1)$ -associative (binary) algebras: we consider the relations defining the Σ_n -module $\mathcal{A}ss(n)$ of the quadratic operad $\mathcal{A}ss$ and define, as above, an algebra with a multiplication which is a bilinear map μ (nonassociative), satisfying the previous relations where μ occurs $n-1$ times in each term of the relations. This algebra will be called $(n-1)$ -associative (binary) algebra. If the Lie bracket of a algebra \mathfrak{g} is also $(n-1)$ -associative, we prove a similar way than for the cubic associative case that \mathfrak{g} is a nilpotent Lie algebra of nilindex $n-1$.
- (2) There exists another notion of associativity for n -ary algebras (an n -ary algebra is a vector space with a multiplication which is an n -linear map), the total associativity. For example, a totally associative 3-ary algebra has a ternary multiplication, denoted xyz , satisfying the relation:

$$(xyz)tu = x(yzt)u = xy(ztu)$$

for any x, y, z, t, u . The corresponding operad is studied in [12] and [15]. Let \mathfrak{g} be a Lie algebra. We have the notion of Lie triple product given by $[[x, y], z]$. If we consider the vector space \mathfrak{g} provided with the 3-ary product given by the Lie triple product, then \mathfrak{g} is a 3-Lie algebra ([5]). Let us suppose now that the Lie triple bracket of \mathfrak{g} is a totally associative product. This implies

$$[[[[X, Y], Z], T], U] = [[X, [[Y, Z], T]], U] = [[X, Y], [[Z, T], U]].$$

But

$$\begin{aligned} [[X, Y], [[Z, T], U]] &= -[[[Z, T], U], [X, Y]] \\ &= [X, [Y, [[Z, T], U]]] + [Y, [[[Z, T], U], X]] \\ &= [[[[Z, T], U], Y], X] - [[[[Z, T], U], X], Y] \\ &= [[Z, [[T, U], Y]], X] - [[Z, [[T, U], X]], Y]. \end{aligned}$$

We deduce

$$\begin{aligned} [[X, [[Y, Z], T]], U] &= [[Z, [[T, U], Y]], X] - [[Z, [[T, U], X]], Y] \\ &= 2^5[[X, [[Y, Z], T]], U] - 2^5[[X, [[Y, Z], U]], T] \end{aligned}$$

Then

$$(2^5 - 1)[[X, [[Y, Z], T]], U] = 2^5[[X, [[Y, Z], U]], T].$$

This implies

$$[[X, [[Y, Z], T]], U] = 0 = [[[[X, Y], Z], T], U] = [[X, Y], [[Z, T], U]].$$

The Lie algebra is 4-step nilpotent.

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